General Relativity and Einstein Equivalence Principle in Non-Inertial and Rotating Frames

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Abstract:

We clearly prove by means of the mathematical perturbation theory the equivalence between a non-inertial uniformly rotating frame and a gravitational field taken as temporally uniform or uniform to first approximation as by Albert Einstein. Our physical and mathematical approach consequently yields Newton's theory of gravity as first approximation. A comparison between Einstein geometrization of gravity and Cartan's is presented. The equivalence principle in its different formulations is explained and related experimental results are shown.

Keywords:

Gravity; Born-Langevin metric; weak equivalence principle; strong equivalence principle; pseudo-Riemannian geometry; curved spacetime; Cartan geometrization

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1. Nonlocal Gravity and Einstein Equivalence Principles

In pseudo-Riemannian geometry, there must exist in the neighborhood of any point P in curved spacetime a system of Cartesian coordinates X^μ of origin P such that the squared spacetime distance takes the form

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu . \qquad (1)$$

This is the mathematical statement and expression of Einstein's equivalence principle[20], where gravity as curvature of spacetime vanishes with respect to a local Cartesian Lorentzian inertial frame defined by X^μ , and where ds is the line element in Minkowski flat spacetime defined by Lorentz or Minkowski metric tensor

$[\eta_{\mu\nu}] = \mathrm{diag}(1, -1, -1, -1)$ (2) or $[\eta_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$. (3)

As a consequence, spacetime is everywhere Lorentzian and locally flat or Minkowskian , and the path of a free massive test body in the local vicinity of a point P with respect to the local inertial frame X^μ is a local-geodesic straight line given by

$$\frac{d^2 X^\mu}{d\tau^2} = 0 \, , \qquad (4)$$

which is nothing but Newton's 1st law in flat spacetime, where the laws of physics are the laws of Einstein's special relativity, analogously to Newton's first law in classical mechanics where a body not subjected to any external force moves in a straight line at constant speed. Thus, locally, where gravity vanishes, even the orbit followed by a satellite in spacetime around the Earth is as straight as possible[1]. But in general relativity spacetime is curved due to gravity and vice versa, whether intrinsically or extrinsically with respect to a non-inertial reference frame, thus flatness of spacetime in Eq. (1) necessitates the Cartesian vanishing of the Riemann curvature tensor, i.e.,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \, , \qquad (5)$$

which, by definition[10][28][56][29], is realized by or due to the presence of a uniform gravitational field or, equivalently or indistinguishably, a uniformly accelerated coordinate system or frame of reference in the absence of gravity, yielding in both cases, when derived, a Newtonian, static or stationary, homogeneous gravitational field, taken to be in the z −direction, and whose derived metric is given by Kottler-Wittaker metric of squared line element[10][28] (6)

$$-ds^2 = dx^2 + dy^2 + \frac{dz^2}{1+2gz} - (1+2gz)dt^2$$

or Møller metric of squared line element

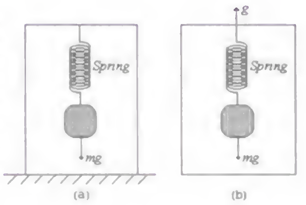
$$-ds^2 = dx^2 + dy^2 + dz^2 - (1+2gz)^2 dt^2 , \qquad (7)$$

which reduces, to a first approximation, in case of a static or stationary weak gravitational field and the non-relativistic limit, where a test body's velocity is small compared to that of light, to Einstein metric[6][10][28][35]

$$-ds^2 = dx^2 + dy^2 + dz^2 - (1+2gz)\, dt^2 , \qquad (8)$$

resulting in an (equivalent) gravitational potential $\phi = gz$ and a gravitational force F $= mg$, where z is the distance away from the centre of the non-inertial uniformly accelerated observer (t, x, y, z) defined at every spacetime point or event P and g is the rest or proper (equivalent gravitational) acceleration constant measured by an accelerometer or in the instantaneous Lorentzian inertial rest frame of the accelerated observer defined by X^{μ} of Eq. (1) and located or centered at $z = 0$. The accelerated observer's non-inertial coordinate system can be materialized as an elevator as shown in Fig.(1).[13]

Fig.1 Two equivalent elevators: (a) an elevator at rest on the earth's surface; (b) an elevator in deep space rising with an acceleration g. Identical objects hang from identical springs, each of which is stretched to the same degree. Each duplicates the physical properties of the other. Observers inside the elevators cannot distinguish between the two possibilities.



As a consequence, an accelerometer reads or measures zero in case of free fall since the proper acceleration vanishes, resulting in a local inertial free-float frame defined by Eq.(1) and with respect to which its freely falling observer and nearby objects float. In Einstein's general relativity, a free-float frame is a local inertial frame moving with constant velocity. An example of such a local inertial frame is a spacecraft in orbit around the Earth. Another example is an elevator for which all cables have broken so that it is freely falling. As a consequence, if an astronaut in the orbiting spacecraft takes an object and leaves it with zero velocity, it stays with zero velocity. This is why the astronauts experience weightlessness. If a person in a freely falling elevator takes an object and leaves it at rest, it stays at rest. Also the person in the elevator experiences weightlessness. Thus, both the astronaut and the person are, within a small enough region of spacetime where gravity is (temporally) uniform, in an inertial frame even though they are both accelerated. The rest or proper acceleration g as measured by the instantaneous comoving inertial frame in the absence of gravity can be set equal to 9.8 m.s^{-2}, which is the gravitational acceleration near or on Earth's surface, in such a way that an accelerometer on Earth's surface reads an upward gravitational acceleration equal to 9.8 m.s^{-2} due to Earth's gravity. On or near Earth's surface, the gravitational field is uniform, weak , Newtonian, and can be described by the spacetime metric of Eq. (6), Eq.(7),or Eq.(8). The local non-inertial coordinates of Eq. (6), (7), and (8), where t is the proper time, are fixed to the uniformly accelerated observer and stand for the non-inertial rest frame of the uniformly accelerated observer and also for the observer's comoving Minkowskian instantaneous inertial rest frame as described by Eq. (1) and centered at $z = 0$, since at every point of spacetime the uniformly accelerated non-inertial frame coincides with its instantaneous comoving Lorentz inertial frame also defined by Eq. (1). For $g = 0$ or $z = 0$, the local non-inertial coordinates of Eqs. (6), (7), and (8) reduce to the locally Minkowskian Cartesian comoving instantaneous inertial frame X^{μ} defined by Eq. (1), which is also the free-float rest frame of the accelerated observer. Metrics of Eqs. (7) and (8), which are approximately or to a first approximation Lorentzian inertial frames, represent Minkowski spacetime as seen by non-inertial observers and yield, by means of the Poincaré coordinate transformations[1][10][11][29][36]

$$\begin{aligned} t' &= (z + 1/g)\sinh(gt), \\ x &= x', \\ y &= y', \\ z' &= (z + 1/g)\cosh(gt), \end{aligned} \tag{9}$$

a Cartesian Minkwoski metric defined by

$$-ds'^2 = dx'^2 + dy'^2 + dz'^2 - dt'^2, \tag{10}$$

where (t', x', y', z') stand for an inertial frame of reference with respect to which the observer accelerates in the z-direction with a four-acceleration whose invariant length is the proper acceleration g and whose spatial acceleration is given by

$$a = \gamma^{-3} g, \tag{11}$$

which in case of Eq. (8) where $v \ll$ c reduces to g. The metric of Eq. (6) also yields Minkowski metric by means of the following coordinate transformations[10]

$$\begin{aligned} t' &= (2z + 1/g)^{1/2}\sinh(gt), \\ x &= x', \\ y &= y', \\ z' &= (2z + 1/g)^{1/2}\cosh(gt). \end{aligned} \tag{12}$$

For the components of Riemann tensor given by Eq. (5) to vanish, the first and the second derivates of the metric tensor must be set equal to zero when it is written as a perturbation of Minkowski metric by means of Taylor-series expansion around P with respect to the local Cartesian inertial coordinates X^{μ} .i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + (\partial_{\sigma} g_{\mu\nu})_P\, X^{\sigma} + \tfrac{1}{2}(\partial_{\sigma}\partial_{\rho} g_{\mu\nu})_P X^{\sigma} X^{\rho} + \cdots, \tag{13}$$

where Einstein summation convention is applied. Consequently, the metric tensor reduces to the diagonal Lorentzian Minkowski metric. On the other hand, in differential geometry, the local neighborhood of a point P on a curved surface can be approximated by a plane, thus similarly, curved spacetime at any point P can also be locally approximated , to a first approximation or to first-order Taylor approximation, by a flat spacetime without being strictly flat as by Eq. (1) by simply setting

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \qquad \text{and} \qquad (\partial_{\sigma} g_{\mu\nu})_P = 0\,, \tag{14}$$

with

$$\tfrac{1}{2}(\partial_{\sigma}\partial_{\rho} g_{\mu\nu})_P X^{\sigma} X^{\rho} \neq 0\,, \tag{15}$$

and

$$\left|\, \tfrac{1}{2}(\partial_{\sigma}\partial_{\rho} g_{\mu\nu})_P X^{\sigma} X^{\rho} \,\right| \ll 1\,, \tag{16}$$

which can be neglected. Thus, the size of the above second derivate determines the region over which

$$g_{\mu\nu} = \eta_{\mu\nu} + \tfrac{1}{2}(\partial_{\sigma}\partial_{\rho} g_{\mu\nu})_P X^{\sigma} X^{\rho} \approx \eta_{\mu\nu}\,, \tag{17}$$

or, equivalently,

$$ds^2 \approx \eta_{\mu\nu}\, dX^{\mu}\, dX^{\nu}, \tag{18}$$

is valid, to a first approximation as by Albert Einstein[3] or to first-order Taylor approximation, also known as first approximation in numerical analysis. A local inertial frame by means of (1) or (18) defines a (nearly) flat Minkowskian tangent spacetime at every point-event P of a curved spacetime resulting in a (roughly) local vanishing of gravity. Physically, locality assigned to the squared Minkowskian line element in Eq. (1) or Eq.(18) means a small enough region of spacetime about the point in question where the gravitational field is strictly or roughly constant, to a first approximation, where the third term on the right of Eq. (13) is neglected as by Eq.(17). Therefore, in a small enough region of spacetime around any spacetime point P where geometry is Minkowskian (to first approximation), general relativity describing gravity as geometry in terms of a spacetime-curvature metric tensor reduces not only to the geometrical theory of special relativity with local Lorentzian inertial frames (to first approximation) but also to Newton's classical gravity with uniform gravitational field (to first approximation) as by Eqs.(6), (7), and (8), where both Newton's equivalence principle between the passive gravitational mass and the inertial mass, also called Eötvös law, and Einstein's weak equivalence principle between acceleration and gravitation, at least to a first approximation as by Einstein[3], are valid. As a consequence, according to Einstein equivalence principle, a local reference frame relative to which gravity vanishes is inertial and given by Eq.(1) ,and, additionally, a local reference frame relative to which a uniform gravitational field emerges is non-inertial and given by Eqs. (6), (7), or (8). The third term on the right of Eq. (13) represents tidal gravitational effects that cannot produce a Newtonian uniform gravitational field and by which Minkowski metric gets perturbed so that gravity becomes present or manifests itself as a curvature of spacetime due to the non-homogeneity of the gravitational field, which is in a sufficiently small region of spacetime vanishes or is neglected. On the other hand, the linear Newton's theory of gravitation can be derived as a first or linear approximation, according to Einstein, from the non-linear Einstein's general relativity by simply perturbing Minkowski metric $\eta_{\mu\nu}$ by means of a Hermitian or symmetric metric tensor $h_{\mu\nu}$ only in its first order , without resorting to Taylor series and without involving the second derivative of the metric $g_{\mu\nu}$, yielding Eq.(8).

2. Free Fall and Equivalence Principles

A test body or an observer freely falling in the local neighborhood of a point-event P occupied by a uniform gravitational field is uniformly accelerated, and its acceleration, according to Einstein's weak equivalence principle, induces an additional and equivalent gravitational field whose magnitude is equal to the real gravitational field but of opposite direction and by

which gravity vanishes with respect to the local (instantaneous) observer's inertial rest frame, which is indeed the local free-float inertial frame defined by Eq. (1) and to which the accelerated non-inertial observer's frame is momentarily equivalent and also with respect to which the accelerated observer's frame is at rest and additionally with which the accelerated non-inertial observer's frame momentarily coincides. But a local inertial frame is equivalent to and stands for Einstein's gravity-free cabin, thus, the observer and nearby objects flow inside the gravity-free cabin provided that the cabin occupies a region of spacetime where gravity is uniform,i.e., where Galileo's equivalence principle which states that all objects freely fall with the same uniform or constant acceleration is valid, and, vice versa consequently , where Newton's equivalence principle stating that gravitational mass equals inertial mass is also valid, which is itself a consequence of the Einstein's weak equivalence principle where a uniform gravitational field is equivalent to a uniform acceleration. Since physics is simple only when analyzed locally, as by Einstein, it is much easier to work the physics of an accelerated observer by replacing it by a continuous succession or sequence of infinite locally rectilinearly or translatory comoving inertial frames, which in case of rotation or orbital motion are locally non-rotating, taking into account that when twisting from one local inertial frame to a different one simultaneity changes. Additionally, it is worthy to note that in general relativity where spacetime is gravitationally curved, there are local inertial frames but no global ones, which prevents from setting up a global Cartesian Minkowskian inertial frame. Therefore, a global non-inertial reference frame viewing spacetime as gravitationally curved can be replaced by an infinite set of local inertial frames such that an accelerated observer in free fall moves through a series of local inertial frames where the laws of the geometrical theory of special relativity are all valid since spacetime is locally flat. Consequently, at each instant, the free-float rest frame of the accelerated freely falling observer is a Lorentz inertial frame centered on the accelerated observer and stands for a freely falling comoving and non-rotating cabin or elevator in a sufficiently small region of spacetime within which the gravitational field can be considered uniform in both direction and magnitude. As the freely falling observer's velocity changes at an (infinitesimally) later instant, local physical observation is handed over to a new Lorentz inertial frame also centered on the accelerated observer and commoving with it at its new velocity. A local inertial frame often stands for a gravity-free lab cabin and to which corresponds a locally comoving and non-rotating orthonormal Cartesian basis of unit vectors called natural tetrad .In general, the gravitational field is not uniform, which is the case of the gravitational field of the Sun, extending radially outward with an attractive effect as shown in Fig.(2).[41] But, for an infinitesimally small enough region of spacetime, the Sun's gravitational field can be considered temporally uniform, both in magnitude and direction, so that the special theory of relativity becomes locally valid resulting in a local Lorentz inertial frame with respect to which gravity vanishes. Consequently, the Earth, which is in its orbital motion is freely falling due to the non-uniform Sun's gravitational field, is equivalent to a local-inertial-frame cabin only for sufficiently small regions of spacetime where the Sun's gravity is (approximately) uniform. Thus, locally, where gravity or curvature vanishes and the laws of physics are the laws of special relativity, a NASA astronaut's accelerometer reads zero as the astronaut orbits the Earth since he or she is weightless due to Einstein's equivalence principle over every small enough region of spacetime where gravity is (approximately) constant or uniform. An observer freely falling in a (nearly) uniform gravitational field, whether Earthwards or inside a spacecraft orbiting the Earth, carries or inhabits a local inertial frame where gravity is eliminated.

Fig.2 Infinitesimally small four-dimensional volume in a gravitational field.

In Newtonian mechanics, gravity also vanishes in a sufficiently small freely falling elevator, defined to be the free-fall observer's local rest frame. In general relativity, Einstein equivalence principle

generalizes this classical locality to include all physical laws of special relativity.

3. Geodesic Equation in Curved Spacetime

However, we wish to work in a single, arbitrary, and general coordinate system with respect to which the freely falling observer accelerates, and which covers all the pseudo-Riemannian spacetime and locally reduces at every point to Eq.(1) or Eq.(18). This can be achieved by maximizing the spacetime distance or proper time between two endpoints of any path, thus minimizing the spatial distance. In general relativity, the spacetime path length or distance is given by

$$s_{AB} = \int_A^B ds = \int_A^B \frac{ds}{d\lambda} d\lambda = \int_A^B \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} d\lambda, \tag{19}$$

where λ is an affine parameter, chosen to be the proper time in case of a massive test body, whilst the integrand

$$L = \frac{ds}{d\lambda} = \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}, \tag{20}$$

is the relativistic Lagrangian. Thus, maximizing s_{AB} yields the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \tag{21}$$

or equivalently

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \tag{22}$$

where the new relativistic Lagrangian is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu. \tag{23}$$

We have

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2}\dot{x}^\rho \dot{x}^\nu \partial_\mu g_{\rho\nu}, \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\rho\nu}\dot{x}^\rho \frac{\partial}{\partial \dot{x}^\mu}\dot{x}^\nu = g_{\rho\nu}\dot{x}^\rho \delta^\nu_\mu = g_{\rho\mu}\dot{x}^\rho,$$

and

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = g_{\rho\mu}\ddot{x}^\rho + \dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} \tag{25}$$

$$= g_{\rho\mu}\ddot{x}^\rho + \frac{1}{2}(\dot{x}^\rho \dot{x}^\nu \partial_\nu g_{\rho\mu} + \dot{x}^\nu \dot{x}^\rho \partial_\rho g_{\nu\mu}),$$

thus Eq. (22) becomes

$$g_{\mu\rho}\ddot{x}^\rho + \frac{1}{2}(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\rho\nu})\dot{x}^\nu \dot{x}^\rho = 0 \tag{26}$$

or

$$g_{\mu\rho}\ddot{x}^\rho + \Gamma_{\mu\nu\rho}\dot{x}^\rho \dot{x}^\nu = 0, \tag{27}$$

which by multiplying by $g^{\lambda\mu}$ becomes

$$\ddot{x}^\lambda + \Gamma^\lambda_{\nu\rho}\dot{x}^\rho \dot{x}^\nu = 0, \tag{28}$$

which is the geodesic equation describing the free motion of a passively massive test body in a pseudo-Riemannian geometry or, equivalently, in a gravitationally curved spacetime whose actively massive or energetic source is the affine connections $\Gamma^\lambda_{\nu\rho}$. In fact, in Einstein's general relativity, an object in free fall is considered intrinsically free and its trajectory or worldline is a geodesic, defined as the path followed by a non-accelerating particle or body and which is locally the shortest path. Plugging the two equations of (14) into (28), which is the case of Minkowskian constant metric components, the metric's first and second partial derivatives vanish and therefore both the connection coefficients and Riemann curvature tensor vanish, and the geodesic equation becomes simply Eq. (1) with a uniform-equivalent-gravitational-acceleration source treated as the zeroth order of the unperturbed metric , which is Minkowski metric as described by Eq.(10). Eqs.(19) up to (28) , in case of constant metric-tensor components, are a clear mathematical proof of the pseudo-Riemannian geometrical property defining the mathematical statement of Einstein equivalence principle resulting in Eq.(1).The geodesic equation (28) can also be written as

$$a^\lambda = 0. \tag{29}$$

which is Newton's first law in curved spacetime, where the term on the right of Eq. (29) is the tensorial intrinsic or absolute acceleration. Thus, in general relativity, gravity becomes part of the fabric of spacetime and manifests itself beyond locality as a curvature in spacetime, unlike Newtonian mechanics where gravity is an external force of attraction between masses.

4. Einstein Equivalence Principle in Rotating Frames

While the Schwarzschild metric, which is the first exact solution of Einstein's field equations, yields the equations describing the motion of a test body in curved spacetime of a

vacuum gravitational field outside of and caused by non-rotating or slowly spinning spherical bodies such as the Earth, neutron stars, and black holes, the Born-Langevin metric is the correct and appropriate spacetime metric used to describe the motion of stationary observers attached to a uniformly rotating rings or to a Born-rigid uniformly rotating discs or bodies where geometry is non-Euclidean with physical significance, and it is given in Cartesian coordinates by the coefficients of the squared line element

$$ds^2 = [c^2 - \omega^2(x^2+y^2)]dt^2 + 2\omega y dt dx - 2\omega x dt dy - dx^2 - dy^2 - dz^2, \quad (30)$$

where (t, x, y, z) stand for a non-inertial rotating frame of reference S. Under the coordinate transformations

$$\begin{aligned} T &\equiv t, \\ X &\equiv x\cos\omega t - y\sin\omega t, \\ Y &\equiv x\sin\omega t + y\cos\omega t, \\ Z &\equiv z, \end{aligned} \quad (31)$$

Eq.(30) becomes

$$ds^2 = c^2 d\tau^2 = c^2 dT^2 - dX^2 - dY^2 - dZ^2, \quad (32)$$

which is the square of the line element defining Minkwoski flat spacetime in inertial Cartesian coordinates (T, X, Y, Z). The non-inertial system of coordinates (t, x, y, z) rotates with respect to the inertial system of coordinates S_{inertial} defined by Eq.(31) at a uniform angular speed ω and both origins coincide. The coordinate transformations given by (30) and the resulting line element of Eq. (31) are valid for any two points over the whole curved manifold described by Born-Langevin metric, resulting in the vanishing of the affine connections $\Gamma^{\lambda}_{\nu\rho}$ everywhere. Thus, spacetime as described by Langevin metric is intrinsically flat but extrinsically or externally curved and its geometry is intrinsically and locally Minkowskian but not extrinsically since the rotating coordinates (t, x, y, z) are not Minkowskian Cartesian inertial coordinates, resulting in the non-vanishing of the components of Riemann tensor. The equations of free motion of a test body or observer with respect to the non-inertial rotating system of coordinates (t, x, y, z) in Langevin-metric spacetime as defined by Eq. (30) can be derived by means of the relativistic Lagrangian

$$L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}, \quad (33)$$

which in case of Eq. (30) becomes

$$L = [c^2 - \omega^2(x^2+y^2)]\dot{t}^2 + 2\omega y\dot{x}\dot{t} - 2\omega x\dot{y}\dot{t} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2, \quad (34)$$

which satisfies the Euler-Lagrange equations of motion

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0, \quad (35)$$

where

$$\dot{x}^{\mu} \equiv dx^{\mu}/d\sigma, \quad (36)$$

and σ is an affine parameter, taken as the proper time in case of massive test bodies.

For $x^1 = x$, one gets

$$\frac{\partial L}{\partial x} = -2\omega^2 x\dot{t}^2 - 2\omega\dot{y}\dot{t} \quad (37)$$

and

$$\frac{\partial L}{\partial \dot{x}} = 2\omega y\dot{t} - 2\dot{x}, \quad (38)$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial \dot{x}} = 2\omega\dot{y}\dot{t} - 2\ddot{x} + 2\omega y\ddot{t}. \quad (39)$$

Substituting Eqs.(37) and (39) into Eq.(35), one gets

$$\ddot{x} - 2\omega\dot{y}\dot{t} - \omega^2 x\dot{t}^2 - \omega y\ddot{t} = 0. \quad (40)$$

For $x^2 = y$, one obtains

$$\ddot{y} - \omega^2 y\dot{t}^2 + 2\omega\dot{x}\dot{t} + \omega x\ddot{t} = 0, \quad (41)$$

while for $x^3 = z$, we have

$$\ddot{z} = 0. \quad (42)$$

For $x^0 = t$, one gets

$$\frac{\partial L}{\partial t} = 0, \quad (43)$$

$$\frac{\partial L}{\partial \dot{t}} = 2c^2\dot{t} - 2\omega^2 x^2\dot{t} - 2\omega^2 y^2\dot{t} + 2\omega y\dot{x} - 2\omega x\dot{y},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial \dot{t}} = 2c^2\ddot{t} - 2\omega^2x^2\ddot{t} - 4\omega^2x\dot{x}\dot{t} - 2\omega^2y^2\ddot{t} - 4\omega^2y\dot{y}\dot{t} + 2\omega y\ddot{x} - 2\omega x\ddot{y} = 0. \quad (44)$$

Multiplying both sides of Eq.(40) by $2\omega y$ and both sides of Eq.(41) by $2\omega x$ and substituting into Eq. (44) , one obtains

$$\ddot{t} = 0, \quad (45)$$

and the equations of motion can accordingly be written as

$$\begin{aligned} \ddot{t} &= 0, \\ \ddot{x} - \omega^2 x\dot{t}^2 - 2\omega\dot{y}\dot{t} &= 0, \\ \ddot{y} - \omega^2 y\dot{t}^2 + 2\omega\dot{x}\dot{t} &= 0, \\ \ddot{z} &= 0. \end{aligned} \quad (46)$$

But $\ddot{t}$ implies that $\dot{t}$ is constant, i.e.,

$$\frac{dt}{d\tau} = \text{constant}, \tag{47}$$

in case of a massive test body. Thus, the equations of motions of (46) can be written as

$$\begin{aligned} d^2x/dt^2 - \omega^2 x - 2\omega\, dy/dt = 0, \\ d^2y/dt^2 - \omega^2 y + 2\omega\, dx/dt = 0, \\ d^2z/dt^2 = 0. \end{aligned} \tag{48}$$

In the absence of real gravity and any other external field, we expect to get the same equations of motion by considering the non-inertial rotating system of coordinates, by means of its Born-Langevin spacetime metric tensor, equivalent to a real gravitational field[8] or, in other words, to be locally equivalent to a gravitational field taken to be uniform (or homogeneous) as by Eq. (7) or uniform (or homogeneous) to a first approximation as by Eq. (8). In general relativity, gravitation is the effect of the departure of a given metric $g_{\mu\nu}$ from Minkwoski metric $\eta_{\mu\nu}$ and "first approximation", according to Einstein, means a small tensor quantity $h_{\mu\nu}$ compared to 1 by which $\eta_{\mu\nu}$ gets perturbed, where $h_{\mu\nu}$ is the dimensionless gravitational metric field by which gravitational effects are produced. Thus, a homogeneous, to a first approximation, gravitational metric field, causing a metrical curvature of spacetime, can be realized by means of (49)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1,$$

taking into account that inhomogeneity resulting in gravitational tidal effects is excluded, since a Newtonian uniform gravitational field cannot produce tidal effects and vice versa, thus the Hermitian or symmetric tensor $h_{\mu\nu}$ does not involve the second derivative of the metric tensor $g_{\mu\nu}$. Equation (49) is simply the well-established mathematical perturbation theory around Minkowski spacetime. In fact, since the Minkowski metric $\eta_{\mu\nu}$ is a known and exact solution, the perturbation theory can be applied yielding

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon\, h^{\mu\nu} + \epsilon^2 h^{\mu\lambda} h_{\lambda}{}^{\nu} + \dots \tag{50}$$

But Newtonian gravity is linear since if we add two masses, the forces they exert on a test body or particle add up as well. Thus, taking into account the smallness of the perturbation $|h_{\mu\nu}| \ll 1$, all higher order terms than the first-order term $h^{\mu\nu}$ or $h_{\mu\nu}$ are neglected and Eq.(50) becomes

$$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon\, h^{\mu\nu}. \tag{51}$$

Additionally, it is clear from Eq.(51) that ϵ is just a fictitious parameter introduced within the formalism of the perturbation theory and can be set equal to 1 , thus Eq.(51) becomes

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{52}$$

or

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{53}$$

The worldline of a passively slow-moving massive test body freely falling under gravity or equivalently in curved spacetime is given by the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0, \tag{54}$$

where (55)

$$\Gamma^\mu{}_{\nu\sigma} = \tfrac{1}{2}\eta^{\mu\rho}(\partial_\nu h_{\sigma\rho} + \partial_\sigma h_{\nu\rho} - \partial_\rho h_{\nu\sigma}),$$

and (56)

$$dx^i/dt \ll c\,(i = 1, 2, 3).$$

By shifting from the proper time τ to the coordinate time t with $x^0 = ct$, Eq.(54) becomes (57)

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = h(t)\frac{dx^\mu}{dt}$$

with (58)

$$h(t) \equiv -\frac{d^2t}{d\tau^2}\left(\frac{dt}{d\tau}\right)^{-2} = \frac{d^2\tau}{dt^2}\left(\frac{d\tau}{dt}\right)^{-1},$$

where the spatial part of Eq.(57) can be written as (59)

$$\frac{1}{c^2}\frac{d^2x^i}{dt^2} + \Gamma^i_{00} + 2\Gamma^i_{0j}\left(\frac{1}{c}\frac{dx^j}{dt}\right) + \Gamma^i_{jk}\left(\frac{1}{c}\frac{dx^j}{dt}\right)\left(\frac{1}{c}\frac{dx^k}{dt}\right) = \frac{1}{c}h(t)\left(\frac{1}{c}\frac{dx^i}{dt}\right).$$

In case of a stationary spacetime metric such as the case of Born-Langevin metric defined by Eq. (30) all the derivatives $\partial_0 h_{\mu\nu} = 0$, and so

$$\begin{aligned} \Gamma^i_{00} &= \tfrac{1}{2}\eta^{i\rho}(\partial_0 h_{0\rho} + \partial_0 h_{0\rho} - \partial_\rho h_{00}) \\ &= -\tfrac{1}{2}\eta^{ij}\partial_j h_{00} = \tfrac{1}{2}\delta^{ij}\partial_j h_{00}. \end{aligned} \tag{60}$$

Also

$$\begin{aligned} \Gamma^i_{0j} &= \tfrac{1}{2}\eta^{i\rho}(\partial_0 h_{j\rho} + \partial_j h_{0\rho} - \partial_\rho h_{0j}) \\ &= -\tfrac{1}{2}\delta^{ik}(\partial_j h_{0k} - \partial_k h_{0j}). \end{aligned} \tag{61}$$

On the other hand, using the mathematical definition of the proper time τ

$$c^2 d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu, \tag{62}$$

we obtain

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{c^2} g_{\mu\nu} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}, \tag{63}$$

which, taking account of Eqs.(53) and (56), becomes

$$d\tau/dt = (1 + h_{00})^{1/2} = 1 + \tfrac{1}{2}h_{00}, \tag{64}$$

yielding

$$d^2\tau/dt^2 = \tfrac{1}{2}ch_{00,0} \tag{65}$$

and

$$\frac{1}{c}h(t) = \tfrac{1}{2}h_{00,0}(1 - \tfrac{1}{2}h_{00}) = \tfrac{1}{2}h_{00,0}, \tag{66}$$

by using Eq.(58). Subsequently, $h(t) = 0$ in case of a stationary or static metric such as Born-Langevin stationary metric. It follows that the right-hand side of Eq.(59) is zero. Additionally, the last term on the left side of Eq.(59) is negligible as by (56). Consequently Eq.(59) becomes

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j(\tfrac{1}{2}c^2h_{00}) + c\delta^{ik}(\partial_j h_{0k} - \partial_k h_{0j})\frac{dx^j}{dt}, \tag{67}$$

where the components of the dimensionless gravitational metric perturbation $h_{\mu\nu}$ can be obtained from Eq. (30) and are written as

$$[h_{\mu\nu}] \equiv \begin{bmatrix} -\omega^2(x^2+y^2)/c^2 & \omega y/c & -\omega x/c & 0 \\ \omega y/c & 0 & 0 & 0 \\ -\omega x/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{68}$$

Both Eqs. (67) and (68) yield

$$\begin{aligned} d^2x/dt^2 - \omega^2 x - 2\omega\, dy/dt &= 0, \\ d^2y/dt^2 - \omega^2 y + 2\omega\, dx/dt &= 0, \\ d^2z/dt^2 &= 0, \end{aligned} \tag{69}$$

which are exactly the same equations of motion as those of (48), where the rotating frame S was not regarded as gravity, and can be written in 3-vector notation with respect to the rotating frame as

$$d^2\mathbf{r}/dt^2 = -\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2\,\boldsymbol{\omega}\times(d\mathbf{r}/dt), \tag{70}$$

where

$$\mathbf{r} = (x, y, z) \text{ and } \boldsymbol{\omega} = (0, 0, \omega). \tag{71}$$

$|\mathbf{r}|$ is the distance from the observer to the origin of the rotating frame. Since the origins of S_{inertial} and S coincide ,

$$\mathbf{r}_{\text{inertial}} = (X, Y, Z) = \mathbf{r}. \tag{72}$$

The two terms on the right-hand side of Eq. (70) are respectively the classical fictitious centrifugal acceleration and the fictitious Coriolis acceleration, which emerge in rotating frames and are on the same footing as real gravitational fields according to Einstein's equivalence principle in general relativity, whereas in special relativity all Lorentz inertial frames are equivalent. However, for an observer at rest in the rotating frame, both fictitious forces are real.[21]Physics of non-inertial frames regarded as gravitational fields is described by Einstein's general relativity.

For a test body observer at rest with respect to the rotating frame S, the Coriolis acceleration, which is a linear velocity-dependent term, vanishes and Eq. (70) becomes

$$\begin{aligned} d^2\mathbf{r}/dt^2 &= -\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \\ &= -\,(\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega} + \omega^2\mathbf{r}. \\ &= \mathbf{a}_{\text{centrifugal}}\,. \end{aligned} \tag{73}$$

Thus, unlike the Coriolis acceleration, the centrifugal acceleration does not depend on the velocity of the observer but only on his position in the rotating frame, and it comes only because of the rotation of the non-inertial frame and can be derived as the gradient of a centrifugal potential associated to the test observer such that

$$\mathbf{a}_{\text{centrifugal}} = -\nabla V \tag{74}$$

with

$$V = -\frac{(\boldsymbol{\omega}\times\mathbf{r})^2}{2}. \tag{75}$$

Additionally, we assume that the observer is in an equatorial plane such as that of the (nearly spherical) Earth without being in contact with the ground and in the absence of gravity. Thus, the first term on the right-hand side of Eq. (73) vanishes. Consequently, Eq. (73) becomes

$$\begin{aligned} d^2\mathbf{r}/dt^2 &= +\,\omega^2\mathbf{r} \\ &= \mathbf{a}_{\text{rotation}}, \end{aligned} \tag{76}$$

which is a centrifugal, center-fleeing, and fictitious pseudo-acceleration, pointing radially outward. According to Einstein equivalence principle, the centrifugal acceleration locally over small regions of spacetime where gravity is sufficiently (roughly) uniform induces an outward equivalent gravitational field that is indistinguishable from a (nearly) uniform gravitational such as that

of the Earth. Thus, as by Einstein, a test object at rest with respect to the rotating frame is in a gravitational field of scalar potential given by Eq.(75), which in case of Born-Langevin metric yields

$$\phi = \frac{-\omega^2\left(x^2+y^2\right)}{2} \tag{77}$$

or, equatorially,

$$\phi = \frac{-\omega^2 r^2}{2}. \tag{78}$$

As by Born-Langevin, metric, a proper clock at rest in the rotating frame beats more slowly than a coordinate clock at the same spacetime location according to

$$d\tau = \sqrt{1-(\omega r/c)^2}dt\,. \tag{79}$$

As a consequence, a signal of frequency f_1 transmitted by a proper clock at a radial distance r_1 will be shifted to a new frequency f_2 when received by a proper clock at a radial distance r_2 as follows

$$\frac{f_2}{f_1} = \frac{\sqrt{1-(\omega r_1/c)^2}}{\sqrt{1-(\omega r_2/c)^2}}. \tag{80}$$

or, by substituting Eq.(78) into Eq.(79),

$$\frac{f_2}{f_1} = \frac{\sqrt{1+2\phi_1/c^2}}{\sqrt{1+2\phi_2/c^2}}. \tag{81}$$

In case the receiver is at the origin of S_{inertial} or S , r_2 = 0, and Eq.(80) or Eq.(81) yields a redshift expressed as

$$\frac{f_2-f_1}{f_1} = \frac{-(\omega r_1)^2}{2c^2}, \tag{82}$$

where $\omega r_1 \ll c$.

Non-inertial observers at rest in the rotating frame interpret the shift as caused by a change in gravitational potential due to the presence of an induced or fictitious gravitational field, unlike an inertial observer at rest in S_{inertial} who interprets the frequency shift as a second order Doppler shift due to time dilation, which can be calculated by applying Lorentz transformation in rotating cylindrical coordinates[57][58] between the inertial frame and a momentarily co-moving frame centered at a point on the rotation path, resulting in Eq.(79).On the other hand, in order to restore external field effects, the observer moving in uniform circular motion with respect to the inertial frame S_{inertial} has a centripetal acceleration given by

$$\mathbf{a}_{\text{inertial}} = -\omega^2\mathbf{r}\,, \tag{83}$$

which is a centripetal, center-seeking, and real acceleration, pointing radially inward. It follows that both the centripetal and centrifugal accelerations are of equal magnitudes and opposite directions. To restore gravity, we set the net centripetal acceleration equal to the gravitational acceleration, i.e.,

$$\mathbf{a}_{\text{inertial}} = -\mathbf{g}\,, \tag{84}$$

which is in fact Galileo equivalence principle and valid only in case of a (temporally) uniform gravitational field. Equation (84) results in a centripetal real or non-fictitious gravitational potential given by

$$\Phi = \frac{\omega^2 r^2}{2}, \tag{85}$$

which is the opposite of the centrifugal gravitational potential given by Eq.(78). By restoring the centripetal acceleration taken to be the gravitational field, the new acceleration of the observer in the observer's non-inertial rotating rest frame S previously given by Eqs. (70) in the absence of the external gravity becomes **0**, i.e.,

$$\begin{aligned}\mathbf{a}_{\text{non-inertial}} &= \mathbf{a}_{\text{inertial}} + \mathbf{a}_{\text{centrifugal}}\\ &= -\mathbf{g} + \omega^2\mathbf{r}\\ &= \mathbf{0}\,,\end{aligned} \tag{86}$$

since the gravitational centripetal acceleration balances out the centrifugal acceleration, which clearly implies that the physical effects of a real gravitational field and those due to an acceleration or rotation field of a non-inertial or rotating frame are equivalent, resulting in a local rest frame with no shift and no gravity and with respect to which objects are weightless and behave as they would in an inertial frame. Thus, as by Albert Einstein, a centrifugal field and a gravitational field are indistinguishable, and a uniformly rotating frame with a centrifugal field as described by Eq. (76) is equivalent to an inertial frame with a gravitational field as described by Eq. (84). Actually, the gravitational field near the surface of the rotating Earth is a mixture of two types of equivalent gravitational fields: the centrifugal gravitational field due to the rotation of the Earth and the Earth's gravitational field caused by its mass since anything that has mass or energy has gravity. Both types of gravity yield the same physical laws, resulting in what is referred to as the principle of equivalence. It must be emphasized that Einstein's equivalence principle between acceleration or rotation of a non-inertial frame and gravitation is valid, due to

gravitational tidal effects, only in spacetime regions where gravity is uniform. Locally, in every small enough region of spacetime, gravity can be considered uniform. As a consequence, in outer space, when a spacecraft orbits the Earth, the centrifugal gravity counterbalances Earth's gravity resulting in zero gravity and astronauts locally experience weightlessness and float with nearby objects.

Artificial gravity can be generated in outer space by means of a spinning cylindrical space station in such a way that, with respect to a NASA astronaut, the centripetal acceleration cannot be distinguished from a real gravitation thanks to Einstein's equivalence principle by which the centripetal acceleration felt by the inhabitants of the space station give the same physical and geometrical effects as a gravitational field of equivalent strength.

5. Einstein's Metrical Geometrization of Newton's Theory of Gravity and Cartan's Geometrized Gravity

By omitting the second term on the right side of Eq. (67) and multiplying its both sides by the inertial mass of a test body, one obtains

$$m\frac{d^2x^i}{dt^2} = -m\delta^{ij}\partial_j(\tfrac{1}{2}c^2h_{00}) \tag{87}$$

or, in 3-vector notation,

$$m\, d^2\mathbf{r}/dt^2 = -m\nabla V \tag{88}$$

or

$$m\mathbf{a} = \mathbf{F} = -m\nabla V, \tag{89}$$

where

$$h_{00} = 2V/c^2. \tag{90}$$

The right-hand side of Eq. (88) is a gradient vector field to which corresponds a scalar function V. Thus, physically, $\mathbf{F}$ is a conservative force which in case of gravity is the gravitational force, yielding a scalar Newtonian gravitational potential

$$V = -GM/r. \tag{91}$$

Therefore, Eq. (88) is nothing but Newton's second law of motion where the external force is the gravitational force. Additionally, setting the right side of Eq. (88) equal to zero yields Newton's first law in classical mechanics. Thus, using the perturbation theory around Minkowski spacetime, the description of gravity as spacetime curvature tends to Newtonian gravity by means of

$$g_{00} = 1 - \frac{2GM}{c^2r} \tag{92}$$

Hence, Newton's theory of gravity can be described by a spacetime metric defined by a squared line element of the form

$$-ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{93}$$

Near the Earth's surface, the gravitational field is uniform and Eq. (89) can be written as

$$V = -\frac{GM_\oplus}{R_\oplus + z}, \tag{94}$$

where $R_\oplus$ and $M_\oplus$ are the radius and mass of the Earth, respectively , and z is the height.

Using Maclaurin series, Eq. (91) can be written as an expansion up to first-order approximation in z in the form

$$V = -\frac{GM_\oplus}{R_\oplus} + \frac{GM_\oplus}{R_\oplus^2}z. \tag{95}$$

Defining the gravitational potential energy as a relative difference in potential energy between two heights, the gravitational potential for a constant gravitational force can be defined by the second term on the right of Eq. (95) so that Eq. (95) is reduced to

$$\phi = gz, \tag{96}$$

where

$$g = GM_\oplus/R_\oplus^2, \tag{97}$$

which is approximately 9.8 m/s^2.

Consequently, using Eq.(96) and setting the speed of light $c = 1$, the squared line element of Eq.(93) can be written as

$$-ds^2 = dx^2 + dy^2 + dz^2 - (1+2\phi)dt^2, \tag{98}$$

which is exactly the same to-first-approximation squared line element of Eq. (8), which was derived by setting the components of the Riemann curvature tensor equal to zero and without the implementation of perturbation theory, as Albert Einstein to derive Eq. (93) and consequently Eq. (98). On the other hand, from the Riemann-Christoffel curvature tensor given by Eq. (5) one obtains

$$R_{00} = \partial_0\Gamma^\mu{}_{0\mu} - \partial_\mu\Gamma^\mu{}_{00} + \Gamma^\nu{}_{0\mu}\Gamma^\mu{}_{\nu 0} - \Gamma^\nu{}_{00}\Gamma^\mu{}_{\nu\mu}, \tag{99}$$

where the second term on the right vanishes since the metric is stationary. Additionally, the third and the fourth terms on the right are of second order in $h_{\mu\nu}$, thus can be neglected and Eq.(99) simply becomes

$$R_{00} = -\partial_i \Gamma^i{}_{00}. \tag{100}$$

Substituting Eq.(60) into Eq.(98), one obtains

$$R_{00} = -\tfrac{1}{2}\delta^{ij}\partial_i\partial_j h_{00}\,. \tag{101}$$

Also, by means of Einstein's field equations written as

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}), \tag{102}$$

one gets

$$R_{00} = -\kappa\left(T_{00} - \tfrac{1}{2}Tg_{00}\right), \tag{103}$$

where

$$\kappa = \frac{8\pi G}{c^4} \approx 2.076\,579 \times 10^{-43}\ \mathrm{s^2\ m^{-1}\ kg^{-1}} \tag{104}$$

is Einstein's gravitational constant.

But in case of a weak gravitational field, the matter-energy source of gravity is dust, thus the only non-zero component of the energy-momentum tensor is

$$T = T_{00} = \rho c^2, \tag{105}$$

and since $|h_{\mu\nu}| \ll 1$, $g_{00} \cong 1$.Thus, Eqs.(101),(103), and (105) yield

$$\delta^{ij}\partial_i\partial_j h_{00} = \kappa\rho c^2. \tag{106}$$

Knowing that

$$\delta^{ij}\partial_i\partial_j = \nabla^2, \tag{107}$$

which is the Laplacian, and substituting Eqs. (100), (104), and (107) into Eq. (104), one gets

$$-c^2 R_{00} = \nabla^2\Phi = 4\pi G\rho, \tag{108}$$

which is Poisson's equation in Newtonian gravity.

Equation (108) as well as Eqs. (100) and (60) actually represent the metrical geometrization of Newton's theory of gravity performed by means of perturbing Minkowski metric $\eta_{\mu\nu}$ with a small spacetime metric tensor $h_{\mu\nu}$ that gives gravitational field effects. In 1923, Élie Cartan mimicked as much as possible Albert Einstein in geometrizing Newtonian gravity using the equivalence principle but without metrically perturbed Minkowski metric such that the affine or metric connections are independent of and not written in function of the metric. In fact, unlike the geodesic equation given by Eq. (27) where the affine connections, also known as the Christoffel symbols of the second kind, are derived in function of the metric components $g_{\mu\nu}$, the geodesic equation can be derived such that its affine connections are metric-independent by means of

$$\frac{dU^\alpha}{d\tau} = \frac{d^2\xi^\alpha}{d\tau^2} = 0, \tag{109}$$

which gives the trajectory of a free-motion test passively massive body in a gravitationally curved spacetime with respect to a local Lorentzian inertial frame defined by ξ^α, where

$$U^\alpha \equiv d\xi^\alpha/d\tau \tag{110}$$

is the four-velocity. Using the chain rule of partial derivatives, Eq.(109) can be written as

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu(\tau)}{d\tau}\right) \\ &= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \end{aligned} \tag{111}$$

where x^μ is an arbitrary coordinate system. To remove the factor by which the second derivative of x^μ is multiplied, we multiply Eq. (111) by

$$\partial x^\lambda/\partial\xi^\alpha \tag{112}$$

taking into account the Kronecker delta

$$\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\right)\left(\frac{\partial x^\lambda}{\partial\xi^\alpha}\right) = \delta^\lambda_\mu\,. \tag{113}$$

Thus, Eq. (111) becomes

$$0 = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{114}$$

where the affine connection is defined by

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\,. \tag{115}$$

The geodesic equation of motion given by Eq. (114) can also be written as

$$\frac{du^\lambda}{d\tau} = -\Gamma^\lambda_{\mu\nu}u^\mu u^\nu, \tag{116}$$

where

$$u^\mu \equiv dx^\mu/d\tau \tag{117}$$

is the four-velocity relative to the arbitrary coordinate system x^μ. On the other hand, Newton's second law of motion is

$$\mathbf{F} = m_i\mathbf{a}, \tag{118}$$

which in case of an external gravitational force becomes

$$m_g \mathbf{g} = m_i \mathbf{a}. \tag{119}$$

But according to Newton's equivalence principle, the gravitational mass m_g is equal to the inertial mass m_i, thus Eq.(119) becomes

$$\mathbf{g} = \mathbf{a}, \tag{120}$$

which is Galileo's equivalence principle by which all objects fall with exactly the same acceleration in a uniform gravitational field. It is noteworthy that Newton's equivalence principle can also be derived from Galileo's equivalence principle. Equation (120) can be written as

$$\mathbf{a} = -\boldsymbol{\nabla}\Phi \tag{121}$$

or

$$\frac{\mathrm{d}^2 x^j}{\mathrm{d}t^2} + \frac{\partial \Phi}{\partial x^j} = 0 \tag{122}$$

or else

$$\frac{\mathrm{d}^2 x^j}{\mathrm{d}t^2} + \frac{\partial \Phi}{\partial x^j}\left(\frac{dt}{dt}\right)^2 = 0, \tag{123}$$

where j is a Latin index for spatial coordinates with j =1, 2, 3. Whilst in Einstein's relativity there are two times, the proper time and the coordinate time, in Newtonian mechanics there is only one time which is absolute. Thus, the proper time used as an affine parameter in Eq. (114) is replaced by the Newtonian time. It follows that when comparing Eq. (114) with Eq. (123), one finds

$$\Gamma^j_{00} = \frac{\partial \Phi}{\partial x^j}, \tag{124}$$

where the geometrized or geometric unit system of natural units is used in which c = G = 1.Using the Riemann curvature tensor as defined by Eq. (5) one obtains

$$R^j_{0k0} = \frac{\partial^2 \Phi}{\partial x^j \partial x^k}. \tag{125}$$

For $j = k$, one gets the time-time component of Ricci curvature tensor, i.e.,

$$R_{00} = R^j_{0j0} \tag{126}$$

or

$$R_{00} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \boldsymbol{\nabla}^2 \Phi, \tag{127}$$

where

$$\boldsymbol{\nabla}^2 \Phi = 4\pi G \rho, \tag{128}$$

which is Poisson's equation for Newtonian gravity. Equation (124) with Eq. (127) is a clear geometrical metric-independent description of Newton's theory of gravity as by Cartan.

6. Overview and Experimental Test Results

In Newton's theory of gravity, both Galileo's and Newton's equivalence principles are solely based on experimental results with no a-priori reason and taken as axioms, unlike Einstein's equivalence principle which emerges as a consequence of the pseudo-Riemannian geometrical nature of general relativity. Additionally, unlike classical or Newtonian mechanics where fictitious fields, which arise only in non-inertial or rotating frames, are simply what separate or distinguish inertial frames from non-inertial frames, general relativity interprets fictitious fields as gravitational fields. According to Einstein's equivalence principle, the physical effects of being accelerated in the absence of gravity are indistinguishable from the physical effects of being in a gravitational field, which can be achieved locally. On the other hand, in general relativity, as a pseudo-Riemannian geometrical theory of gravity, spacetime curvature can be either intrinsic or extrinsic, yielding physically a real or apparent gravitational field, respectively, and vice versa. As a consequence, in spite of the equivalence between acceleration and gravitation, two major physically geometrical properties distinguish an equivalent or apparent gravitational field produced by a non-inertial frame and a real or permanent or actual gravitational field. Whereas the latter vanishes at infinity or when the source of spacetime curvature vanishes yielding the flat Minkowsksi metric, which is the case of the Schwarzschild metric when $r \to \infty$ or when $M \to 0$, the former increases indefinitely at large distances and vanishes at the origin of the rotation axis, which is the case of the well-known Born-Langevin metric. The other major difference is that while the spacetime metric that stands for a real gravitational field cannot be transformed to a Cartesian Minkowski metric over the whole manifold by any coordinate transformation even though locally at any spacetime point on can define a Cartesian Lorentz inertial

reference frame as by Eq.(1) where gravity is eliminated, the spacetime metric representing an apparent gravitational field can be transformed to a Cartesian flat metric over all spacetime by means of a special coordinate transformation where all affine connections vanish resulting in the vanishing of gravity as metrical curvature of spacetime. The Equivalence Principle in its weak form neglects self-gravity, which involves the non-linear properties of gravity, and considers that strong and electroweak interactions do not affect the freely falling test body. The WEP incorporates both the equivalence between the passive gravitational mass and the inertial mass, which is a consequence of Einstein weak equivalence principle, and the universality of free fall that all objects freely fall with the same acceleration in a uniform gravitational field, and its validity has been tested greatly many times in terms of Eötvös parameter η with no violation being reported by all experiments so far with an upper limit on η given with 95% confidence[39] as

$$|\eta\,(\mathrm{Au},\mathrm{Al})| < 3\times 10^{-11} \quad (129)$$

and best-laboratory limits also on η given by[40]

$$\eta\,(\mathrm{Be},\mathrm{Ti}) = (0.3\pm 1.8)\times 10^{-13} \quad (130)$$

and

$$\eta\,(\mathrm{Be},\mathrm{Al}) = (-0.7\pm 1.3)\times 10^{-13} \quad (131)$$

with 1σ uncertainty. All numerical results of Eqs. (127), (128), and (129) are not significantly different from zero, thus do not show any lack of equivalence between the passive gravitational mass and the inertial mass. The smaller the value of η, the better the precision and the accuracy of the test. The validity of the WEP was also tested and confirmed by the MICROSCOPE-satellite mission, and the analysis of the first 2017 published data give a limit on η[40] as

$$|\eta\,(\mathrm{Ti},\mathrm{Pt})| = [-1\pm 9\,(\mathrm{stat})\pm 9\,(\mathrm{syst})]\times 10^{-15} \quad (132)$$

with a 1σ uncertainty. When gravitational self-energy is considered, the EP is called the strong equivalence principle by which the path of any freely falling body even with highly measurable self-gravity is locally flat. The SEP is tested either in terms of Nordtvedt parameter η_N or in terms of Damour-Schäfer parameter Δ.The validity of the SEP was tested by the NASA MESSENGER-spacecraft mission, and the 2018 published results confirm that there is no violation of the SEP and are consistent with Einstein's theory of general relativity[37], yielding limits on η_N estimated as

$$\eta_N = (-6.6\pm 7.2)\times 10^{-5} \quad (133)$$

with 1σ uncertainty. The SEP was also tested by tracking a spinning-neutron-star pulsar in a stellar triple system, known as PSR J0337+1715,composed of two white dwarfs and the pulsar, involving the pulsar's strong self-gravity[38]. Einstein's SEP and general relativity pass this strong gravitational-field test with flying colors, and the 2018 published results give an upper limit on Damour-Schäfer parameter Δ estimated as

$$|\Delta| < 2.6\times 10^{-6} \quad (134)$$

with 95% confidence. Taking into account that

$$\Delta = \eta_N E, \quad (135)$$

where $E \cong 0.1$ is the dimensionless or fractional binding energy of the pulsar, and substituting (134) into (135) one obtains the strong-field Nordtvedt parameter

$$|\eta_N| < 2.6\times 10^{-5}. \quad (136)$$

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